

**New Formulation of Matter Effects on Neutrino Mixing and  $CP$  Violation****Zhi-zhong Xing** \**Sektion Physik, Universität München, Theresienstrasse 37A, 80333 Munich, Germany***Abstract**

Within the framework of three lepton families we have derived an exact and compact formula to describe the matter effect on neutrino mixing and  $CP$  violation. This model- and parametrization-independent result can be particularly useful for recasting the fundamental lepton flavor mixing matrix from a variety of long-baseline neutrino experiments.

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Recent observation of the atmospheric and solar neutrino anomalies by the Super-Kamiokande Collaboration has provided robust evidence for massive neutrinos that mix among different flavors [1]<sup>†</sup>. It opens a convincing window for new physics beyond the Standard Model, and has important cosmological implications.

Although the non-accelerator neutrino experiments have yielded some impressive constraints on the parameter space of atmospheric and solar neutrino oscillations, a precise determination of the lepton flavor mixing angles and the leptonic  $CP$ -violating phase(s) has to rely on a new generation of accelerator experiments with very long baselines [4], including the possible neutrino factories [5]. In such long-baseline neutrino experiments the earth-induced matter effects, which are likely to deform the neutrino oscillation behaviors in vacuum and even to fake the genuine  $CP$ -violating asymmetries, must be taken into account. To single out the “true” theory of lepton mass generation and  $CP$  violation depends crucially upon how accurately the fundamental parameters of lepton flavor mixing can be measured and disentangled from the matter effects. It is therefore desirable to explore the *analytical* relationship between the genuine flavor mixing matrix and the matter-corrected one beyond the conventional two-flavor framework [6, 7, 8, 9]. Some attempts of this nature have so far been made [7, 10, 11, 12], but they are subject to specific assumptions, approximations or parametrizations and need be substantially improved.

In this note we present an exact and compact formula to describe the matter effect on lepton flavor mixing and  $CP$  violation within the framework of three lepton families. The result is completely independent of the specific models of neutrino masses and the specific parametrizations of neutrino mixing. Therefore it will be particularly useful, in the long run, to recast the fundamental flavor mixing matrix from the precise measurements of neutrino oscillations in a variety of long-baseline neutrino experiments.

First let us define the  $3 \times 3$  lepton flavor mixing matrix in vacuum to be  $V$ . It links the neutrino mass eigenstates  $(\nu_1, \nu_2, \nu_3)$  to the neutrino flavor eigenstates  $(\nu_e, \nu_\mu, \nu_\tau)$ :

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (1)$$

If neutrinos are massive Dirac fermions,  $V$  can be parametrized in terms of three rotation angles and one  $CP$ -violating phase. If neutrinos are Majorana fermions, however, two additional  $CP$ -violating phases are in general needed to fully parametrize  $V$ . The strength of  $CP$  violation in neutrino oscillations, no matter whether neutrinos are of the Dirac or Majorana type, depends only upon a universal parameter  $\mathcal{J}$  [13]:

$$\text{Im} \left( V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^* \right) = \mathcal{J} \sum_{\gamma, k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk}, \quad (2)$$

where  $(\alpha, \beta, \gamma)$  and  $(i, j, k)$  run over  $(e, \mu, \tau)$  and  $(1, 2, 3)$ , respectively. In the specific models of fermion mass generation  $V$  can be derived from the mass matrices of charged leptons and neutrinos [14]. To test such theoretical models one has to compare their

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<sup>†</sup>Throughout this work the LSND evidence for neutrino oscillations [2], which has not been independently confirmed by other experiments [3], will not be taken into account.

predictions for  $V$  with the experimental data of neutrino oscillations. The latter may in most cases be involved in the potential matter effects and must be carefully handled.

In matter the effective Hamiltonian for neutrinos can be written as [6]

$$\mathcal{H}_\nu = \frac{1}{2E} \left[ V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right], \quad (3)$$

where  $m_i$  (for  $i = 1, 2, 3$ ) denote neutrino masses,  $A = 2\sqrt{2}G_F N_e E$  describes the charged-current contribution to the  $\nu_e e^-$  forward scattering,  $N_e$  is the background density of electrons, and  $E$  stands for the neutrino beam energy. The neutral-current contributions, which are universal for  $\nu_e$ ,  $\nu_\mu$  and  $\nu_\tau$  neutrinos, lead only to an overall unobservable phase and have been neglected. Transforming  $\mathcal{H}_\nu$  from the flavor eigenbasis to the mass eigenbasis, we obtain the effective mass-squared matrix of neutrinos:

$$\Omega_\nu = \begin{pmatrix} m_1^2 + A|V_{e1}|^2 & AV_{e1}^* V_{e2} & AV_{e1}^* V_{e3} \\ AV_{e1} V_{e2}^* & m_2^2 + A|V_{e2}|^2 & AV_{e2}^* V_{e3} \\ AV_{e1} V_{e3}^* & AV_{e2} V_{e3}^* & m_3^2 + A|V_{e3}|^2 \end{pmatrix}, \quad (4)$$

This Hermitian matrix can be diagonalized through a unitary transformation:  $U^\dagger \Omega_\nu U = \text{Diag}\{\lambda_1, \lambda_2, \lambda_3\}$ , where  $\lambda_i$  denote the effective mass-squared eigenvalues of neutrinos. Explicitly one finds <sup>‡</sup>

$$\begin{aligned} \lambda_1 &= m_1^2 + \frac{1}{3}x - \frac{1}{3}\sqrt{x^2 - 3y} \left[ z + \sqrt{3(1 - z^2)} \right], \\ \lambda_2 &= m_1^2 + \frac{1}{3}x - \frac{1}{3}\sqrt{x^2 - 3y} \left[ z - \sqrt{3(1 - z^2)} \right], \\ \lambda_3 &= m_1^2 + \frac{1}{3}x + \frac{2}{3}z\sqrt{x^2 - 3y}, \end{aligned} \quad (5)$$

where  $x$ ,  $y$  and  $z$  are given by

$$\begin{aligned} x &= \Delta m_{21}^2 + \Delta m_{31}^2 + A, \\ y &= \Delta m_{21}^2 \Delta m_{31}^2 + A \left[ \Delta m_{21}^2 (1 - |V_{e2}|^2) + \Delta m_{31}^2 (1 - |V_{e3}|^2) \right], \\ z &= \cos \left[ \frac{1}{3} \arccos \frac{2x^3 - 9xy + 27A\Delta m_{21}^2 \Delta m_{31}^2 |V_{e1}|^2}{2(x^2 - 3y)^{3/2}} \right] \end{aligned} \quad (6)$$

with  $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$  and  $\Delta m_{31}^2 \equiv m_3^2 - m_1^2$ . Note that the columns of  $U$  are the eigenvectors of  $\Omega_\nu$  in the vacuum mass eigenbasis. After a lengthy calculation, we arrive at the elements of  $U$  as follows:

$$U_{ii} = \frac{N_i}{D_i}, \quad U_{ij} = \frac{A}{D_j} (\lambda_j - m_k^2) V_{ei}^* V_{ej}, \quad (7)$$

where  $(i, j, k)$  run over  $(1, 2, 3)$  with  $i \neq j \neq k$ , and

$$\begin{aligned} N_i &= (\lambda_i - m_j^2) (\lambda_i - m_k^2) - A \left[ (\lambda_i - m_j^2) |V_{ek}|^2 + (\lambda_i - m_k^2) |V_{ej}|^2 \right], \\ D_i^2 &= N_i^2 + A^2 |V_{ei}|^2 \left[ (\lambda_i - m_j^2)^2 |V_{ek}|^2 + (\lambda_i - m_k^2)^2 |V_{ej}|^2 \right]. \end{aligned} \quad (8)$$

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<sup>‡</sup>The expressions of  $\lambda_i$  in Ref. [7] have apparent printing errors, while those given in Ref. [10] depend upon a specific parametrization of  $V$ .

One may check that  $A = 0$  leads to  $\lambda_i = m_i^2$  and  $D_i^2 = N_i^2$ , and then  $U$  becomes the unity matrix.

It should be noted that the unitary matrix  $U$  transforms the neutrino mass eigenstates in matter to those in vacuum. Therefore the lepton flavor mixing matrix in matter, denoted by  $V^m$ , is a product of the lepton flavor mixing matrix in vacuum ( $V$ ) and the matter-to-vacuum transformation matrix  $U$ :  $V^m = VU$ . The elements of  $V^m$  turn out to be

$$V_{\alpha i}^m = \frac{N_i}{D_i} V_{\alpha i} + \frac{A}{D_i} V_{ei} \left[ (\lambda_i - m_j^2) V_{ek}^* V_{\alpha k} + (\lambda_i - m_k^2) V_{ej}^* V_{\alpha j} \right], \quad (9)$$

where  $\alpha$  runs over  $(e, \mu, \tau)$  and  $(i, j, k)$  over  $(1, 2, 3)$  with  $i \neq j \neq k$ . Obviously  $A = 0$  leads to  $V_{\alpha i}^m = V_{\alpha i}$ . This exact and compact formula shows clearly how the flavor mixing matrix in vacuum is corrected by the matter effects. Instructive analytical approximations can be made for Eq. (9), once the hierarchy of neutrino masses is experimentally known or theoretically assumed.

The result obtained above is valid for neutrinos interacting with matter. As for antineutrinos, the corresponding formula for the flavor mixing matrix in matter can straightforwardly be obtained from Eq. (9) through the replacements  $V \Rightarrow V^*$  and  $A \Rightarrow -A$ .

With the help of Eq. (9) we now calculate the universal  $CP$ -violating parameter in matter, defined as  $\mathcal{J}_m$  through

$$\text{Im} \left( V_{\alpha i}^m V_{\beta j}^m V_{\alpha j}^{m*} V_{\beta i}^{m*} \right) = \mathcal{J}_m \sum_{\gamma, k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk}, \quad (10)$$

where  $(\alpha, \beta, \gamma)$  and  $(i, j, k)$  run over  $(e, \mu, \tau)$  and  $(1, 2, 3)$ , respectively. The unitarity of  $V^m$  allows us to simplify the lengthy calculation and arrive at an instructive relationship between  $\mathcal{J}_m$  and  $\mathcal{J}$ :

$$\mathcal{J}_m = \mathcal{J} \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \quad (11)$$

The same result has been obtained by Harrison and Scott using the Jarlskog determinant of lepton mass matrices, which is invariant for neutrinos in vacuum and in matter [15]. Eq. (11) indicates that the matter contamination to  $CP$ - and  $T$ -violating observables is in general unavoidable. However,  $T$  violation is expected to be less sensitive to matter effects than  $CP$  violation, since the former is associated only with either neutrinos ( $+A$ ) or antineutrinos ( $-A$ ) while the latter is related to both of them.

To be more explicit we calculate the conversion probabilities of  $\nu_\alpha$  (or  $\bar{\nu}_\alpha$ ) to  $\nu_\beta$  (or  $\bar{\nu}_\beta$ ) neutrinos in matter. We obtain

$$\begin{aligned} P_m(\nu_\alpha \rightarrow \nu_\beta) &= -4 \sum_{i < j} [\text{Re}(V_{\alpha i}^m V_{\beta j}^m V_{\alpha j}^{m*} V_{\beta i}^{m*}) \sin^2 \Delta_{ij}] + 8 \mathcal{J}_m \prod_{i < j} \sin \Delta_{ij}, \\ P_m(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) &= -4 \sum_{i < j} [\text{Re}(\tilde{V}_{\alpha i}^m \tilde{V}_{\beta j}^m \tilde{V}_{\alpha j}^{m*} \tilde{V}_{\beta i}^{m*}) \sin^2 \tilde{\Delta}_{ij}] - 8 \tilde{\mathcal{J}}_m \prod_{i < j} \sin \tilde{\Delta}_{ij}, \end{aligned} \quad (12)$$

where  $(\alpha, \beta)$  run over  $(e, \mu)$ ,  $(\mu, \tau)$  or  $(\tau, e)$ ;  $\tilde{V}_{\alpha i}(A) \equiv V_{\alpha i}(-A)$ ,  $\tilde{\Delta}_{ij}(A) \equiv \Delta_{ij}(-A)$ , and  $\tilde{\mathcal{J}}_m(A) \equiv \mathcal{J}_m(-A)$ ; and  $\Delta_{ij} \equiv 1.27(\lambda_i - \lambda_j)L/E$  with  $L$  the distance between the

production and interaction points of  $\nu_\alpha$  (in unit of km) and  $E$  the neutrino beam energy (in unit of GeV). The probabilities of  $\nu_\beta \rightarrow \nu_\alpha$  and  $\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha$  transitions can be read off from Eq. (11) with the replacements  $\mathcal{J}_m \Rightarrow -\mathcal{J}_m$  and  $\tilde{\mathcal{J}}_m \Rightarrow -\tilde{\mathcal{J}}_m$ , respectively.

One can then define the  $CP$ - and  $T$ -violating asymmetries as

$$\begin{aligned}\mathcal{A}_{CP} &= \frac{P_m(\nu_\alpha \rightarrow \nu_\beta) - P_m(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)}{P_m(\nu_\alpha \rightarrow \nu_\beta) + P_m(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)}, \\ \mathcal{A}_T &= \frac{P_m(\nu_\alpha \rightarrow \nu_\beta) - P_m(\nu_\beta \rightarrow \nu_\alpha)}{P_m(\nu_\alpha \rightarrow \nu_\beta) + P_m(\nu_\beta \rightarrow \nu_\alpha)}.\end{aligned}\tag{13}$$

Note that  $\mathcal{A}_T = \mathcal{A}_{CP}$  holds in vacuum (i.e.,  $A = 0$ ), as a consequence of  $CPT$  invariance. Any discrepancy between these two observables will definitely measure the matter effects in long-baseline neutrino experiments.

Finally let us give a numerical illustration of the matter-induced corrections to the flavor mixing matrix and  $CP$  (or  $T$ ) violation in vacuum. The elements of  $V^m$ , except the Majorana phases, can be completely determined by four rephasing-invariant quantities (e.g., four independent  $|V_{\alpha i}^m|$  or three independent  $|V_{\alpha i}^m|$  plus  $\mathcal{J}_m$ ). As the solar and atmospheric neutrino oscillations in vacuum are essentially associated with the elements in the first row and the third column of  $V$ , it is favored to choose  $|V_{e1}|$ ,  $|V_{e2}|$ ,  $|V_{\mu 3}|$  and  $\mathcal{J}$  as the four basic parameters. For illustration we take  $|V_{e1}| = 0.816$ ,  $|V_{e2}| = 0.571$ ,  $|V_{\mu 3}| = 0.640$ , and  $\mathcal{J} = \pm 0.020$  for neutrinos and antineutrinos<sup>§</sup>. Such a choice is consistent with the CHOOZ experiment [17], the large-angle MSW solution to the solar neutrino problem [18], and a nearly maximal mixing in the atmospheric neutrino oscillation. The relevant neutrino mass-squared differences are typically taken to be  $\Delta m_{21}^2 = 5 \cdot 10^{-5} \text{ eV}^2$  and  $\Delta m_{31}^2 = 3 \cdot 10^{-3} \text{ eV}^2$  [1, 18]. To calculate the  $CP$ - and  $T$ -violating asymmetries in the long-baseline neutrino experiments, we assume a constant earth density profile and take  $A = 2.28 \cdot 10^{-4} \text{ eV}^2 E / [\text{GeV}]$  [12]. We also choose the baseline length  $L = 732 \text{ km}$ , corresponding to a neutrino source at Fermilab pointing toward the Soudan mine or that at CERN toward the Gran Sasso underground laboratory. Using these inputs as well as Eqs. (9)–(12), we first take  $(\alpha, \beta) = (e, \mu)$  and compute the asymmetries  $\mathcal{A}_{CP}$  and  $\mathcal{A}_T$  changing with  $E$  in the range  $2 \text{ GeV} \leq E \leq 30 \text{ GeV}$ . Then we compute the ratios  $|V_{\alpha i}^m|/|V_{\alpha i}|$  and  $\mathcal{J}_m/\mathcal{J}$  as functions of the matter parameter  $A$ , instead of  $E$ , in the range  $10^{-7} \text{ eV}^2 \leq A \leq 10^{-2} \text{ eV}^2$ . The numerical results are shown in Figs. 1 and 2.

We observe that matter effects can be significant for the elements in the first and the second columns of  $V$ , if  $A \geq 10^{-5} \text{ eV}^2$ . In comparison, the magnitudes of  $|V_{e3}|$ ,  $|V_{\mu 3}|$  and  $|V_{\tau 3}|$  may be drastically enhanced or suppressed only for  $A > 10^{-3} \text{ eV}^2$ . The neutrinos are relatively more sensitive to the matter effects than the antineutrinos.

The magnitude of  $\mathcal{J}_m$  decreases, when the matter effect becomes significant (e.g.,  $A \geq 10^{-4} \text{ eV}^2$ ). However, this does not imply that the  $CP$ - or  $T$ -violating asymmetries in realistic long-baseline neutrino oscillations would be smaller than their values in vacuum. Large matter effects can significantly modify the frequencies of neutrino oscillations

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<sup>§</sup>This specific choice corresponds to  $\theta_{12} \approx 35^\circ$ ,  $\theta_{23} \approx 40^\circ$ ,  $\theta_{13} \approx 5^\circ$ , and  $\delta \approx \pm 90^\circ$  in the PDG-advocated parametrization of  $V$  [16].

and thus enhance (or suppress) the genuine signals of  $CP$  or  $T$  violation. As for the long-baseline neutrino experiment under consideration, the matter-induced effect in the  $T$ -violating asymmetry  $\mathcal{A}_T$  is negligibly small. The matter effect on the  $CP$ -violating asymmetry in vacuum cannot be neglected, but the former is unlikely to fake the latter completely. We confirm numerically that the relationship  $\mathcal{A}_T = \mathcal{A}_{CP}$  in vacuum becomes violated in matter.

If the earth-induced matter effects can well be controlled, it is possible to recast the fundamental flavor mixing matrix  $V$  from a variety of measurements of neutrino oscillations. Such a goal is expected to be reached in the neutrino factories [5].

In summary, we have derived an exact and compact formula to show the analytical relationship between the fundamental neutrino mixing matrix and the matter-corrected one within the framework of three lepton families. This model- and parametrization-independent result can be particularly useful for the study of flavor mixing and  $CP$  violation in the long-baseline neutrino experiments. An extension of the present work, in which the mixing of a sterile neutrino with three active neutrinos can be incorporated, is in progress.

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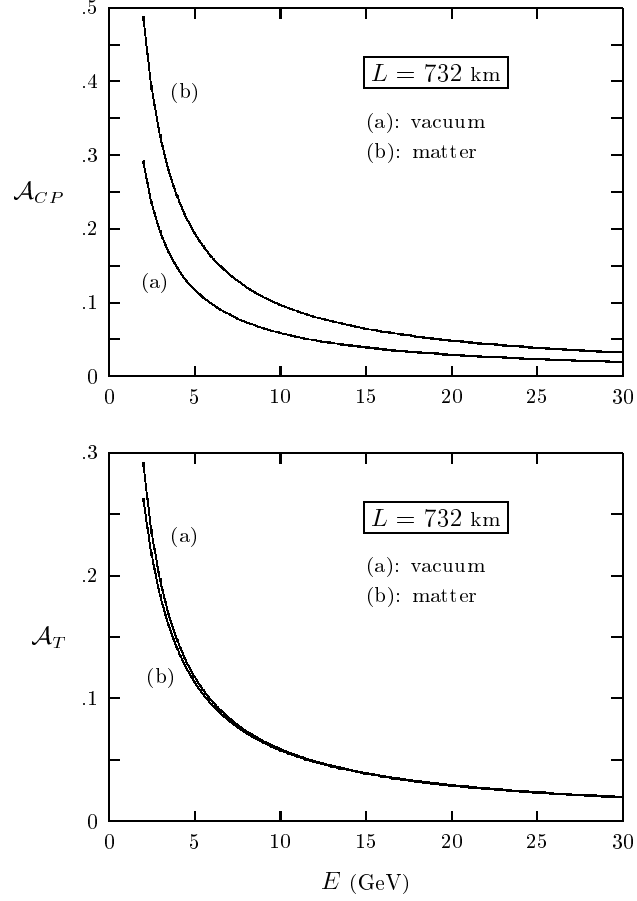


Figure 1: Illustrative plots for matter effects on  $\mathcal{A}_{CP}$  (between  $\nu_e \rightarrow \nu_\mu$  and  $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$  transitions) and  $\mathcal{A}_T$  (between  $\nu_e \rightarrow \nu_\mu$  and  $\nu_\mu \rightarrow \nu_e$  transitions) in a long-baseline neutrino experiment with  $L = 732$  km, where  $\Delta m_{21}^2 = 5 \cdot 10^{-5} \text{ eV}^2$  and  $\Delta m_{31}^2 = 3 \cdot 10^{-3} \text{ eV}^2$  have typically been input.



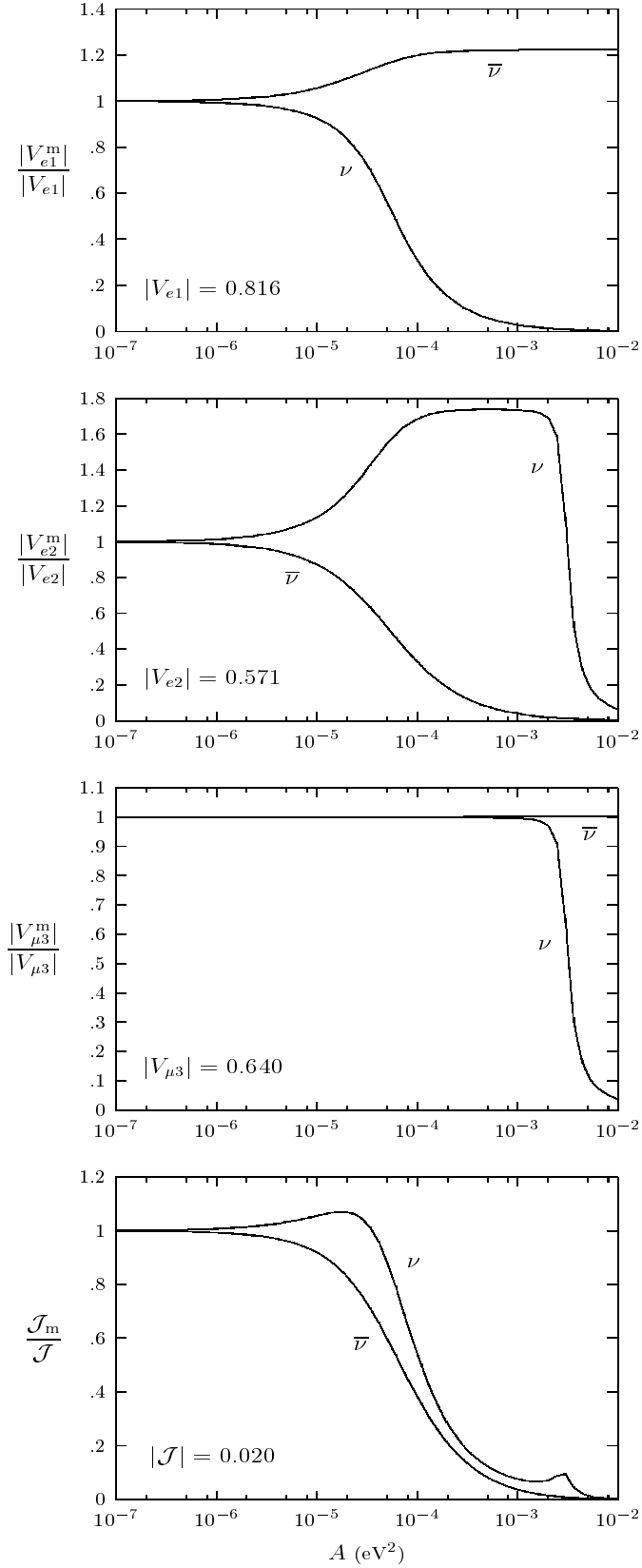


Figure 2: Illustrative plots for matter effects on  $|V_{e1}|$ ,  $|V_{e2}|$ ,  $|V_{\mu 3}|$  and  $\mathcal{J}$  associated with neutrinos (+ $A$ ) and antineutrinos ( $-A$ ), where  $\Delta m_{21}^2 = 5 \cdot 10^{-5} \text{ eV}^2$  and  $\Delta m_{31}^2 = 3 \cdot 10^{-3} \text{ eV}^2$  have typically been input.